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LAGRANGIAN COHERENT STRUCTURES IN FINITE DOMAINS

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ABSTRACT

We develop a finite-domain finite-time Lyapunov exponent (FDFTLE) method to allow Lagrangian Coherent Structure (LCS) extraction from velocity data within limited domains. This removes spurious ridges as seen when trajectories are stopped at the domain boundaries. We find this extension useful in practical applications when LCS are extracted from LIDAR measurements at Hong Kong International Airport and used to determine airflow patterns around the airport. In addition to the FDFTLE method, we have developed a suite of mathematical tools to quantify different types of air motion near the LCS. This allows us to objectively describe the relative motion near LCS.

INTRODUCTION

The use of Lagrangian Coherent Structures (LCS) in the objective, frame-independent identification of transport and mixing structures in nonlinear fluid flows has been a popular trend in recent years [2, 4, 5, 7]. In the computation of the mathematical criteria that signifies LCS, initial conditions are integrated over time using a given velocity field to obtain the Lagrangian trajectory. Certain dynamical properties are evaluated along the trajectories to reveal Lagrangian coherence. For example, the finite-time Lyapunov exponent indicates the amount of stretching of nearby trajectories over a finite time considered [3]. In real applications, as a rule rather than exception, velocity fields are specified on open domains. This poses significant challenge in the computation of LCS when fluid trajectories meet the boundaries and leave the domain, since stopping the trajectories will artificially make the boundaries attractors and repellers, a false structure that is undesired.

One way to mitigate the problem is artificially extending the data to a linear external velocity field. The external velocity is obtained by least square approximation of the given data in L^2 norm while maintaining incompressibility. Velocity data and extrapolation are then connected by a filter function that smoothly

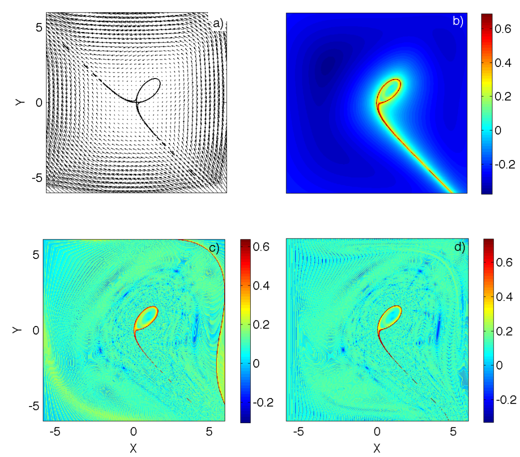

FIGURE 1. Application of the FDFTLE method for an idealized flow field.

connects the interior and exterior of the domain [6]. By this extension, trajectories meeting the domain boundaries can continue to separate at the rate that follows a global, large-scale flow, a most probable fate of a fluid trajectory. Artificial structures are eliminated since nearby trajectories all move in a smooth velocity field. We show one example of the application of the FDFTLE method to a flow of known dynamical boundaries in Fig. 1. The flow field is given by $u = x - y^2; v = -y + x^2$. This autonomous system has one fixed point at the origin, with one branch of stable manifold, one branch of unstable manifold and a homoclinic orbit passing the fixed point. Using velocity data from unlimited range we accurately extract the dynamical boundaries precisely (cf. Fig. 1b). To compare the FDFTLE with a traditional method, we artificially limit the domain and stop trajectories at the boundaries (Fig. 1c) and apply the FDFTLE method (Fig. 1d). Clearly,

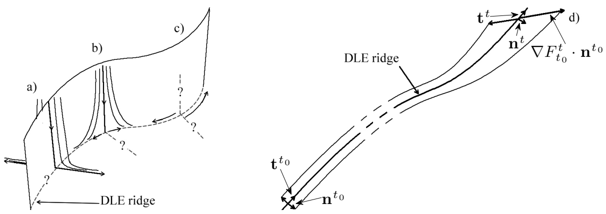

FIGURE 2. Different Lagrangian measures helping the differentiation of flow structures.

there is an artificial ridge in Fig. 1c which is removed in Fig. 1d.

In the particular problem of interest described in [6], which is also the focus of this talk, we are interested in the accurate extraction of LCS in a small observational domain covered by the range limit of the LIght Detection And Ranging equipment (LIDAR) situated at the Hong Kong International Airport. Because of the limiting range, LIDAR would provide a nowcast of flow features that an airplane would experience during the first or last minute of its flight. Using the traditional method of stopping trajectories at the domain boundaries significantly limits the regions of trustworthy structures that can be used by the pilot. As such, the application of the FDFTLE method is desirable in detecting the nonlinearity of the near ground flow.

It is not only the LCS, or the FDFTLE field that is of our interest. Air motion near the FDFTLE can be stretched in several ways: transversal stretching to a ridge line of FDFTLE (hyperbolic, repelling structures), or sliding along a ridge line (parabolic, shear structures). In reality both types of dynamics exist around a LCS. We use a set of measures based on Fenichel numbers [1] to differentiate the LCS and thus are able to tell if the type of air motion we see is dangerous to airplane approaches. The measures can be understood through Fig. 2. Fig. 2a-c indicate the several possible motions that a FTLE ridge can indicate, whereas Fig. 2d indicates how the normal and tangent vectors to the FTLE ridge are used to form different Lagrangian measures.

Finally, in this talk, we use LIDAR measured data to reconstruct a 2D flow field and detect coherent structures in this reconstruction. The extracted LCS is compared to flight data to understand their correlation and help develop algorithms that would generate warnings for airplanes leaving/approaching the HKIA.

MATHEMATICS

We briefly outline the various mathematical equations here. The linear extension technique and the filter function are outlined in [6]. To obtain the FDFTLE field, we compute the flow map $\mathbf{x}(t;\mathbf{x}_0;t_0)$ from the reconstructed velocity

$$\dot{\mathbf{x}} = \mathbf{u}, \tag{1}$$

and the FDFTLE is computed as

$$\sigma(t;t_0,\mathbf{x}_0) = \frac{1}{2|t-t_0|}\ln\left(\left[\frac{\partial \mathbf{x}(t;t_0,\mathbf{x}_0)}{\partial \mathbf{x}_0}\right]^T \frac{\partial \mathbf{x}(t;t_0,\mathbf{x}_0)}{\partial \mathbf{x}_0}\right) \tag{2}$$

An integrated measure of the vertical motion of the airflow is given as

$$\begin{aligned}
DIV_{t_0}^t(\mathbf{x}_0) &= \frac{1}{|t-t_0|}\int_{t_0}^{t}\left[\frac{\partial u(\mathbf{x})}{\partial x}+\frac{\partial v(\mathbf{x})}{\partial y}\right]\mathrm{d}t\,,\\
STR_{\perp} &= \frac{1}{|t-t_0|}\ln\left([\mathbf{n}^t]^T\cdot\nabla F_{t_0}^t\cdot\mathbf{n}^{t_0}\right),\\
STR_{\parallel} &= \frac{1}{|t-t_0|}\ln\left([\mathbf{t}^t]^T\cdot\nabla F_{t_0}^t\cdot\mathbf{t}^{t_0}\right),\\
SHR &= \frac{1}{|t-t_0|}\ln\left([\mathbf{t}^t]^T\cdot\nabla F_{t_0}^t\cdot\mathbf{n}^{t_0}\right)
\end{aligned} \tag{3}$$

where DIV is the Lagrangian integral of the horizontal divergence along a trajectory, and the horizontal measures indicate Lagrangian versions of normal, tangential stretching and along structure shear.

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